

# Experiments with Tractable Feedback in Robotic Planning under Uncertainty: Insights over a wide range of noise regimes

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**Abstract.** We consider the problem of robotic planning under uncertainty. This problem may be posed as a stochastic optimal control problem, complete solution to which is fundamentally intractable owing to the infamous curse of dimensionality. We report the results of an extensive simulation study in which we have compared two methods, both of which aim to salvage tractability by using alternative, albeit inexact, means for treating feedback. The first is a recently proposed method based on a near-optimal “decoupling principle” for tractable feedback design, wherein a nominal open-loop problem is solved, followed by a linear feedback design around the open-loop. The second is Model Predictive Control (MPC), a widely-employed method that uses repeated re-computation of the nominal open-loop problem during execution to correct for noise, though when interpreted as feedback, this can only be said to be an implicit form. We examine a much wider range of noise levels than have been previously reported and empirical evidence suggests that the decoupling method allows for tractable planning over a wide range of uncertainty conditions without unduly sacrificing performance.

**Keywords:** Empirical study, Optimization, Optimal Control

## 1 Introduction

Planning under uncertainty is a central problem in robotics. The space of current methods includes several contenders, each with different simplifying assumptions, approximations, and domains of applicability. This is a natural consequence of the fact that the challenge of dealing with the continuous state, control and observation space problems, for non-linear systems and across long-time horizons with significant noise, and potentially multiple agents, is fundamentally intractable.

Model Predictive Control is one popular means for tackling optimal control problems [13, 19]. The MPC approach solves a finite horizon “deterministic” optimal control problem at every time step given the current state of the process, performs only the first control action and then repeats the planning process at the next time step. In terms of computation, this can be a costly endeavor and,

when a stochastic control problem is well approximated by the deterministic problem (when the noise is meager), much of this computation is superfluous.

In this paper we consider the generalization of a recently proposed method [25] that uses a local feedback to control noise induced deviations from the deterministic (that we term the *nominal*) trajectory. When the deviation is too large for the feedback to manage, replanning is triggered and it computes a fresh nominal. Otherwise, the feedback tames the perturbations during execution and no computation is expended in replanning. Put another way, the method decouples feedback and planning/nominal control but will fall back to replanning when perturbations are excessive. Thus, by considering every deviation to necessitate replanning, this approach will essentially reduce to MPC itself.

We present an empirical investigation of this decoupling approach, exploring dimensions that are important in characterizing its performance—key among these being the triggering of replanning. Hence, the primary focus of the study is on understanding the performance across a wide range of noise conditions with comparison to the “gold standard” of MPC. Figure 1 gives an overall summary of the paper’s findings: the areas under the respective curves give the total computational resources consumed—the savings by the decoupling method over MPC are seen to be considerable.

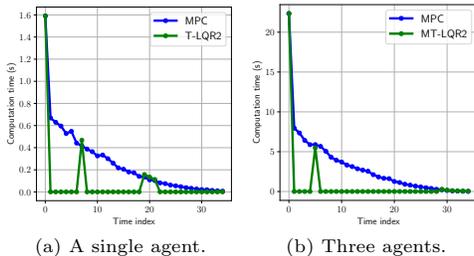


Fig. 1: Computation time expended by MPC (in blue) and the decoupling algorithms described herein (in green), at each time step for a sample experiment involving navigation. Both cases result in nearly identical motions by the robot. The peaks in T-LQR2 and MT-LQR2 happen only when replanning takes place. Computational effort decreases for both methods because the horizon diminishes as the agent(s) reach their goals. (To relate to subsequent figures: noise parameter  $\epsilon = 0.4$  and the replan threshold = 2% of cost deviation.)

### 1.1 Related Work

Robotic planning problems under uncertainty can be posed as a stochastic optimal control problem that requires the solution of an associated Dynamic Programming (DP) problem, however, as the state dimension increases, the computational complexity goes up exponentially [4], Bellman’s infamous “curse of dimensionality”. There has been recent success using sophisticated (Deep) Reinforcement Learning (RL) paradigm to solve DP problems, where deep neural networks are used as the function approximators [2, 10, 11, 22, 23], however, the training time required for these approaches is still prohibitive to permit real-time robotic planning that is considered here.

In the case of continuous state, control and observation space problems, the Model Predictive Control [13, 19] approach has been used with a lot of success

in the control system and robotics community. For deterministic systems, the process results in solving the original DP problem in a recursive online fashion. However, stochastic control problems, and the control of uncertain systems in general, is still an unresolved problem in MPC. As succinctly noted in [13], the problem arises due to the fact that in stochastic control problems, the MPC optimization at every time step cannot be over deterministic control sequences, but rather has to be over feedback policies, which is, in general, difficult to accomplish since a tractable parametrization of such policies to perform the optimization over, is, in general, unavailable. Thus, the tube-based MPC approach, and its stochastic counterparts, typically consider linear systems [7, 14, 20] for which a linear parametrization of the feedback policy suffices but the methods become intractable when dealing with nonlinear systems. In recent work, we have introduced a “decoupling principle” that allows us to tractably solve such stochastic optimal control problems in a near optimal fashion, with applications to highly efficient RL and MPC implementations [17, 25]. However, this prior work required a small noise assumption. In this work, we relax this small noise assumption to show, via extensive empirical evaluation, that even when the noise is not small, a *replan-when-necessary* modification of the decoupled planning approach, akin to event-triggered MPC [9, 12], suffices to keep the planning computationally efficient while retaining performance comparable to MPC. We note that event-triggered MPC inherits the same issues mentioned above with respect to the stochastic control problem, and consequently, the techniques are only tractable for linear systems. *Lest it seem that we are being unduly critical of MPC, that is definitely not our intention: we believe that MPC type replanning is unavoidable in uncertain systems, instead we additionally believe that such replanning can be substantially reduced utilizing decoupling while tractably and rigorously extending MPC to stochastic systems, i.e., the decoupled approach is not competition, but rather complimentary, to MPC. Please also see the deeper historical context to this discussion at the end of Section 3.1, after we have presented the basic near-optimality result.*

The problem of multiple agents further and severely compounds the planning problem since now we are also faced with the issue of a control space that grows exponentially with the number of agents in the system. Moreover, since the individual agents never have full information regarding the system state, the observations are partial. Furthermore, the decision making has to be done in a distributed fashion which places additional constraints on the networking and communication resources. In a multi-agent setting, the stochastic optimal problem can be formulated in the space of joint policies. Some variations of this problem have been successfully characterized and tackled based on the level of observability, in/dependence of the dynamics, cost functions and communications [16, 18, 21]. This has resulted in a variety of solutions from fully-centralized [5] to fully-decentralized approaches with many different subclasses [3, 15].

The major concerns of the multi-agent problem are tractability of the solution and the level of communication required during the execution of the policies.

In this paper, we also consider a generalization of the decoupling principle to a multi-agent, fully observed setting. We show that this leads to a spatial decoupling between agents in that they do not need to communicate for long periods of time during execution. Albeit, we do not consider the problem of when and how to replan in this paper, assuming that there exists a (yet to be determined) distributed mechanism that can achieve this, we nonetheless show that there is a highly significant increase in planning efficiency over a wide range of noise levels.

## 2 Problem Formulation

The problem of robot planning and control under noise can be formulated as a stochastic optimal control problem in the space of feedback policies. We assume here that the map of the environment is known and state of the robot is fully observed. Uncertainty in the problem lies in the system's actions.

### 2.1 System Model:

For a dynamic system, we denote the state and control vectors by  $\mathbf{x}_t \in \mathbb{X} \subset \mathbb{R}^{n_x}$  and  $\mathbf{u}_t \in \mathbb{U} \subset \mathbb{R}^{n_u}$  respectively at time  $t$ . The motion model  $f : \mathbb{X} \times \mathbb{U} \times \mathbb{R}^{n_w} \rightarrow \mathbb{X}$  is given by the equation

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t, \epsilon \mathbf{w}_t); \mathbf{w}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_{\mathbf{w}_t}), \quad (1)$$

where  $\{\mathbf{w}_t\}$  are zero mean independent, identically distributed (i.i.d) random sequences with variance  $\mathbf{\Sigma}_{\mathbf{w}_t}$ , and  $\epsilon$  is a small parameter modulating the noise input to the system.

### 2.2 Stochastic optimal control problem:

The stochastic optimal control problem for a dynamic system with initial state  $\mathbf{x}_0$  is defined as:

$$J_{\pi^*}(\mathbf{x}_0) = \min_{\pi} \mathbb{E} \left[ \sum_{t=0}^{T-1} c(\mathbf{x}_t, \pi_t(\mathbf{x}_t)) + c_T(\mathbf{x}_T) \right], \quad (2)$$

$$s.t. \mathbf{x}_{t+1} = f(\mathbf{x}_t, \pi_t(\mathbf{x}_t), \epsilon \mathbf{w}_t), \quad (3)$$

where:

- the optimization is over feedback policies  $\pi := \{\pi_0, \pi_1, \dots, \pi_{T-1}\}$  and  $\pi_t(\cdot) : \mathbb{X} \rightarrow \mathbb{U}$  specifies an action given the state,  $\mathbf{u}_t = \pi_t(\mathbf{x}_t)$ ;
- $J_{\pi^*}(\cdot) : \mathbb{X} \rightarrow \mathbb{R}$  is the cost function on executing the optimal policy  $\pi^*$ ;
- $c_t(\cdot, \cdot) : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}$  is the one-step cost function;
- $c_T(\cdot) : \mathbb{X} \rightarrow \mathbb{R}$  is the terminal cost function;
- $T$  is the horizon of the problem;
- the expectation is taken over the random variable  $\mathbf{w}_t$ .

### 3 A Decoupling Principle

Now, we give a brief overview of a “decoupling principle” that allows us to substantially reduce the complexity of the stochastic planning problem given that the parameter  $\epsilon$  is small enough. We only provide an outline here and the relevant details can be found in our recent work [25]. We shall also present a generalization to a class of multi-robot problems. Finally, we preview the results in the rest of the paper.

#### 3.1 Near-Optimal Decoupling in Stochastic Optimal Control

Let  $\pi_t(\mathbf{x}_t)$  denote a control policy for the stochastic planning problem above, not necessarily the optimal policy. Consider now the control actions of the policy when the noise to the system is uniformly zero, and let us denote the resulting “nominal” trajectory and controls as  $\bar{\mathbf{x}}_t$  and  $\bar{\mathbf{u}}_t$  respectively, i.e.,  $\bar{\mathbf{x}}_{t+1} = f(\bar{\mathbf{x}}_t, \bar{\mathbf{u}}_t, 0)$ , where  $\bar{\mathbf{u}}_t = \pi_t(\bar{\mathbf{x}}_t)$ . Note that this nominal system is well defined. Further, let us assume that the closed-loop (i.e., with  $\mathbf{u}_t = \pi_t(\mathbf{x}_t)$ ), system equations, and the feedback law are smooth enough that we can expand the feedback law about the nominal as  $\pi_t(\mathbf{x}_t) = \bar{\mathbf{u}}_t + \mathbf{K}_t \delta \mathbf{x}_t + \mathbf{R}_t^\pi(\delta \mathbf{x}_t)$ , where  $\delta \mathbf{x}_t = \mathbf{x}_t - \bar{\mathbf{x}}_t$ , i.e., the perturbation from the nominal,  $\mathbf{K}_t$  is the linear gain obtained by the Taylor expansion about the nominal in terms of the perturbation  $\delta \mathbf{x}_t$ , and  $\mathbf{R}_t^\pi(\cdot)$  represents the second and higher order terms in the expansion of the feedback law about the nominal trajectory. Further we assume that the closed-loop perturbation state can be expanded about the nominal as:  $\delta \mathbf{x}_t = \mathbf{A}_t \delta \mathbf{x}_t + \mathbf{B}_t \mathbf{K}_t \delta \mathbf{x}_t + \mathbf{R}_t^f(\delta \mathbf{x}_t) + \epsilon \mathbf{B}_t \mathbf{w}_t$ , where the  $\mathbf{A}_t$ ,  $\mathbf{B}_t$  are the system matrices obtained by linearizing the system state equations about the nominal state and control, while  $\mathbf{R}_t^f(\cdot)$  represents the second and higher order terms in the closed-loop dynamics in terms of the state perturbation  $\delta \mathbf{x}_t$ . Moreover, let the nominal cost be given by  $\bar{J}^\pi = \sum_{t=0}^T \bar{c}_t$ , where  $\bar{c}_t = c(\bar{\mathbf{x}}_t, \bar{\mathbf{u}}_t)$ , for  $t \leq T-1$ , and  $\bar{c}_T = c_T(\bar{\mathbf{x}}_T, \bar{\mathbf{u}}_T)$ . Further, assume that the cost function is smooth enough that it permits the expansion  $J^\pi = \bar{J} + \sum_t \mathbf{C}_t \delta \mathbf{x}_t + \sum_t \mathbf{R}_t^c(\delta \mathbf{x}_t)$  about the nominal trajectory, where  $\mathbf{C}_t$  denotes the linear term in the perturbation expansion and  $\mathbf{R}_t^c(\cdot)$  denote the second and higher order terms in the same. Finally, define the exactly linear perturbation system  $\delta \mathbf{x}_{t+1}^\ell = \mathbf{A}_t \delta \mathbf{x}_t^\ell + \mathbf{B}_t \mathbf{K}_t \delta \mathbf{x}_t^\ell + \epsilon \mathbf{B}_t \mathbf{w}_t$ . Further, let  $\delta J_1^{\pi, \ell}$  denote the cost perturbation due to solely the linear system, i.e.,  $\delta J_1^{\pi, \ell} = \sum_t \mathbf{C}_t \delta \mathbf{x}_t^\ell$ . Then, the decoupling result states the following [25]:

**Theorem 1.** *The closed-loop cost function  $J^\pi$  can be expanded as  $J^\pi = \bar{J}^\pi + \delta J_1^{\pi, \ell} + \delta J_2^\pi$ . Furthermore,  $\mathbb{E}[J^\pi] = \bar{J}^\pi + O(\epsilon^2)$ , and  $\text{Var}[J^\pi] = \text{Var}[\delta J_1^{\pi, \ell}] + O(\epsilon^4)$ , where  $\text{Var}[\delta J_1^{\pi, \ell}]$  is  $O(\epsilon^2)$ .*

Thus, the above result suggest that the mean value of the cost is determined almost solely by the nominal control actions while the variance of the cost is almost solely determined by the linear closed-loop system. Thus, the decoupling result says that the feedback law design can be decoupled into an open-loop and a closed-loop problem.

*Open-Loop Problem:* This problem solves the deterministic/ nominal optimal control problem:

$$\bar{J} = \min_{\bar{\mathbf{u}}_t} \sum_{t=0}^{T-1} c(\bar{\mathbf{x}}_t, \bar{\mathbf{u}}_t) + c_T(\bar{\mathbf{x}}_T), \quad (4)$$

subject to the nominal dynamics:  $\bar{\mathbf{x}}_{t+1} = f(\bar{\mathbf{x}}_t, \bar{\mathbf{u}}_t)$ .

*Closed-Loop Problem:* One may try to optimize the variance of the linear closed-loop system

$$\min_{\mathbf{K}_t} \text{Var}[\delta J_1^{\pi, \ell}] \quad (5)$$

subject to the linear dynamics  $\delta \mathbf{x}_{t+1}^\ell = \mathbf{A}_t \delta \mathbf{x}_t^\ell + \mathbf{B}_t \mathbf{K}_t \delta \mathbf{x}_t^\ell + \epsilon \mathbf{B}_t \mathbf{w}_t$ . However, the above problem does not have a standard solution but note that we are only interested in a good variance for the cost function and not the optimal one. Thus, this may be accomplished by a surrogate LQR problem that provides a good linear variance as follows.

*Surrogate LQR Problem:* Here, we optimize the standard LQR cost:

$$\delta J_{\text{LQR}} = \min_{\mathbf{u}_t, \mathbf{w}_t} \mathbb{E} \left[ \sum_{t=0}^{T-1} \delta \mathbf{x}_t^\top \mathbf{Q} \delta \mathbf{x}_t + \delta \mathbf{u}_t^\top \mathbf{R} \delta \mathbf{u}_t + \delta \mathbf{x}_T^\top \mathbf{Q}_f \delta \mathbf{x}_T \right], \quad (6)$$

subject to the linear dynamics  $\delta \mathbf{x}_{t+1}^\ell = \mathbf{A}_t \delta \mathbf{x}_t^\ell + \mathbf{B}_t \delta \mathbf{u}_t + \epsilon \mathbf{B}_t \mathbf{w}_t$ . In this paper, this decoupled design shall henceforth be called the trajectory-optimized LQR (T-LQR) design.

**A Historical Context.** The above decoupled design might seem like a perturbation feedback design outlined in classical optimal control texts such as (Ch. 6, [6]) and we are certainly not claiming that we are the first to discover it. However, the perturbation design was always thought to be heuristic and its “goodness” for the stochastic optimal control problem was essentially unexplored. Notable as an exception is the reference [8] that considers the problem of how good the deterministic feedback law is for the stochastic system, which is shown to be  $O(\epsilon^4)$ . However, that paper assumes the availability of the optimal deterministic feedback law which is the solution of the deterministic Hamilton-Jacobi-Bellman (HJB) equation (the DP equation in continuous time problems), which, in itself, is intractable as noted by Fleming as the “practical difficulty” in this work (pgs. 475–476 of [8]). However, MPC, by repeatedly solving the deterministic optimal control problem at every time step, implicitly furnishes the *deterministic feedback law*, and thus, offers the solution to the *practical dilemma* above. The field of MPC, of course, was developed almost two decades after Fleming’s work, while stochastic MPC/ MPC-under-uncertainty was explored only starting at the turn of millennium [13]. Thus, this connection was lost and never really explored in the MPC literature. This connection is critical if we want to *tractably* extend MPC to stochastic systems in a *theoretically justifiable fashion*, in the sense that in much of the stochastic MPC literature, these two aspects are at cross purposes to each other thereby preventing a satisfactory

resolution. Thus, the MPC replanning logic is well justified theoretically, even when applied to a stochastic system.

In fact, with a few further developments, and adaptation of Fleming’s work to discrete time finite horizon problems, and if the linear feedback gain is modified suitably, the perturbation design also becomes  $O(\epsilon^4)$  near-optimal. Due to paucity of space, we postpone this result to a future paper, however, for the sake of completeness and the reader’s benefit, the result is included in the supplementary document. *The ultimate takeaway is that the implicit MPC feedback law is an excellent approximation to the optimal stochastic policy, however, a T-LQR type perturbation feedback design is much cheaper computationally, while retaining identical near-optimality guarantees as MPC.*

### 3.2 Multi-agent setting

Now, we generalize the above result to a class of multi-agent problems. We consider a set of agents that are transition independent, i.e, their dynamics are independent of each other. For simplicity, we also assume that the agents have perfect state measurements. Let the system equations for the agents be given by:  $\mathbf{x}_{t+1}^j = f(\mathbf{x}_t^j) + \mathbf{B}_t^j(\mathbf{u}_t^j + \epsilon \mathbf{w}_t^j)$ , where  $j = 1, 2, \dots, M$  denotes the  $j^{\text{th}}$  agent. (We have assumed the control affine dynamics for simplicity). Further, let us assume that we are interested in the minimization of the joint cost of the agents given by  $\mathcal{J} = \sum_{t=0}^{T-1} c(\mathbf{X}_t, \mathbf{U}_t) + \Phi(\mathbf{X}_T)$ , where  $\mathbf{X}_t = [\mathbf{x}_t^1, \dots, \mathbf{x}_t^M]$ , and  $\mathbf{U}_t = [\mathbf{u}_t^1, \dots, \mathbf{u}_t^M]$  are the joint state and control action of the system. The objective of the multi-agent problem is minimize the expected value of the cost  $\mathbb{E}[\mathcal{J}]$  over the joint feedback policy  $\mathbf{U}_t(\cdot)$ . The decoupling result holds here too and thus the multi-agent planning problem can be separated into an open and closed-loop problem. The open-loop problem consists of optimizing the joint nominal cost of the agents subject to the individual dynamics.

*Multi-Agent Open-Loop Problem:*

$$\bar{\mathcal{J}} = \min_{\bar{\mathbf{U}}_t} \sum_{t=0}^{T-1} c(\bar{\mathbf{X}}_t, \bar{\mathbf{U}}_t) + \Phi(\bar{\mathbf{X}}_T), \quad (7)$$

subject to the nominal agent dynamics  $\bar{\mathbf{x}}_{t+1}^j = f(\bar{\mathbf{x}}_t^j) + \mathbf{B}_t^j \bar{\mathbf{u}}_t^j$ . The closed-loop, in general, consists of optimizing the variance of the cost  $\mathcal{J}$ , given by  $\text{Var}[\delta \mathcal{J}_1^\ell]$ , where  $\delta \mathcal{J}_1^\ell = \sum_t \mathbf{C}_t \delta \mathbf{X}_t^\ell$  for suitably defined  $\mathbf{C}_t$ , and  $\delta \mathbf{X}_t^\ell = [\delta \mathbf{x}_t^1, \dots, \delta \mathbf{x}_t^M]$ , where the perturbations  $\delta \mathbf{x}_t^j$  of the  $j^{\text{th}}$  agent’s state is governed by the decoupled linear multi-agent system  $\delta \mathbf{x}_t^j = \mathbf{A}_t \delta \mathbf{x}_t^j + \mathbf{B}_t^j \delta \mathbf{u}_t^j + \epsilon \mathbf{B}_t^j \mathbf{w}_t^j$ . This design problem does not have a standard solution but recall that we are not really interested in obtaining the optimal closed-loop variance, but rather a good variance. Thus, we can instead solve a surrogate LQR problem given the cost function  $\delta \mathcal{J}_{\text{MTLQR}} = \sum_{t=0}^{T-1} \sum_j \delta \mathbf{x}_t^j \mathbf{Q}^j \delta \mathbf{x}_t^j + \delta \mathbf{u}_t^j \mathbf{R} \delta \mathbf{u}_t^j + \sum_j \delta \mathbf{x}_T^j \mathbf{Q}_f^j \delta \mathbf{x}_T^j$ . Since the cost function itself is decoupled, the surrogate LQR design degenerates into a decoupled LQR design for each agent.

*Surrogate Decoupled LQR Problem:*

$$\delta\mathcal{J}^j = \min_{\mathbf{u}_t^j, \mathbf{w}_t^j} \mathbb{E} \left[ \sum_{t=0}^{T-1} \delta\mathbf{x}_t^j \top \mathbf{Q}^j \delta\mathbf{x}_t^j + \delta\mathbf{u}_t^j \top \mathbf{R} \delta\mathbf{u}_t^j + \delta\mathbf{x}_T^j \top \mathbf{Q}_f^j \delta\mathbf{x}_T^j \right], \text{ subject to the}$$

linear decoupled agent dynamics  $\delta\mathbf{x}_t^j = \mathbf{A}_t \delta\mathbf{x}_t^j + \mathbf{B}_t^j \delta\mathbf{u}_t^j + \epsilon \mathbf{B}_t^j \mathbf{w}_t^j$ .

*Remark 1.* Note that the above decoupled feedback design results in a spatial decoupling between the agents in the sense that, at least in the small noise regime, after their “initial joint plan” is made, the agents never need to communicate with each other in order to complete their missions. However, note that the joint plan requires communication.

### 3.3 Planning Complexity versus Uncertainty

The decoupling principle outlined above shows that the complexity of planning can be drastically reduced while still retaining near optimal performance for sufficiently small noise (i.e., parameter  $\epsilon \ll 1$ ). Nonetheless, the skeptical reader might argue that this result holds only for low values of  $\epsilon$  and thus, its applicability for higher noise levels is suspect. Still, because the result is second order, it hints that near optimality might be over a reasonably large  $\epsilon$ . Naturally, the question is ‘*will it hold for medium to higher levels of noise?*’ We purposely leave the terminology of medium to high noise nebulous but what we mean shall become clear from our experiments.

*Preview of the Results.* In this paper, we illustrate the degree to which the above result still holds when we allow periodic replanning of the nominal trajectory in T-LQR in an event triggered fashion, dubbed T-LQR2. Here, we shall use MPC as a “gold standard” for comparison since the true stochastic control problem is intractable, and owing to Fleming’s result [8], the MPC policy is  $O(\epsilon^4)$  near-optimal when compared to the true stochastic policy. In fact, we can make an identical strong  $O(\epsilon^4)$  claim for T-LQR as well if the *linear feedback gain is designed carefully*, but owing to the paucity of space, testing with this careful feedback design is left to a future paper. Here, we show that though the number of replanning operations in T-LQR2 increases the planning burden over T-LQR, it is still much reduced when compared to MPC, which replans continually. The ability to trigger replanning means that T-LQR2 can always produce solutions with the same quality as MPC, albeit by demanding the same computational cost as MPC in instances when replanning is triggered. But for moderate levels of noise, T-LQR2 can produce comparable quality output to MPC with substantial computational savings.

In the high noise regime, replanning is more frequent but we shall see that there is another consideration at play. Namely, that the effective planning horizon decreases and there seems no benefit in planning all the way to the end rather than considering only a few steps ahead, and in fact, in some cases, it can be

harmful to consider the distant future. Noting that as the planning horizon decreases, planning complexity decreases, this helps recover tractability even in this regime.

Thus, while lower levels of noise render the planning problem tractable due to the decoupling result requiring no replanning, planning under medium noise remains tractable due to only occasional replanning, while for high levels of noise, tractability ensues because the planning horizon should shrink as the uncertainty increases. When noise inundates the system, long-term predictions become so uncertain that the best-laid plans will very likely run awry, and thus, it would be wasteful to invest significant time thinking very far ahead. To examine this somewhat intuitive truth more quantitatively, the parameter  $\epsilon$  will be a knob we adjust, exploring these aspects in the subsequent empirical analysis. *We reiterate that the notion of low, medium and high noise regimes may seem somewhat vague, however, we provide precise definitions of these regimes using our empirical results later in this paper.*

## 4 The Planning Algorithms

The preliminaries and the algorithms are explained below:

### 4.1 Deterministic Optimal Control Problem:

Given the initial state  $\mathbf{x}_0$  of the system, the solution to the deterministic OCP is given by (4), s.t.  $\mathbf{x}_{t+1} = f(\mathbf{x}_t) + \mathbf{B}_t \mathbf{u}_t$ ,  $\mathbf{u}_{\min} \leq \mathbf{u}_t \leq \mathbf{u}_{\max}$ ,  $|\mathbf{u}_t - \mathbf{u}_{t-1}| \leq \Delta \mathbf{u}_{\max}$ . The last two constraint model physical limits that impose upper bounds and lower bounds on control inputs and rate of change of control inputs. The solution to the above problem gives the open-loop control inputs  $\bar{\mathbf{u}}_{0:T-1}$  for the system. For our problem, we take a quadratic cost function for state and control as  $c_t(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{x}_t^T \mathbf{W}^x \mathbf{x}_t + \mathbf{u}_t^T \mathbf{W}^u \mathbf{u}_t$ ,  $c_T(\mathbf{x}_T) = \mathbf{x}_T^T \mathbf{W}_f^x \mathbf{x}_T$ , where  $\mathbf{W}^x$ ,  $\mathbf{W}_f^x \succeq \mathbf{0}$  and  $\mathbf{W}^u \succ \mathbf{0}$ .

### 4.2 Model Predictive Control (MPC):

We employ the non-linear MPC algorithm due to the non-linearities associated with the motion model. The MPC algorithm implemented here solves the deterministic OCP (4) at every time step, applies the control inputs computed for the first instant and uses the rest of the solution as an initial guess for the subsequent computation. In the next step, the current state of the system is measured and used as the initial state and the process is repeated.

### 4.3 Short Horizon MPC (MPC-SH):

We also implement a variant of MPC which is typically used in practical applications where it solves the OCP only for a short horizon rather than the entire horizon at every step. At the next step, a new optimization is solved over the shifted horizon. This implementation gives a greedy solution but is computationally easier to solve. It also has certain advantageous properties in high noise

cases which will be discussed in the results section. We denote the short planning horizon as  $H_c$  also called as the control horizon, upto which the controls are computed.

#### 4.4 Trajectory Optimised Linear Quadratic Regulator (T-LQR):

As discussed in Section 3, stochastic optimal control problem can be decoupled and solved by designing an optimal open-loop (nominal) trajectory and a decentralized LQR policy to track the nominal.

*Design of nominal trajectory:* The nominal trajectory is generated by first finding the optimal open-loop control sequence by solving the deterministic OCP (4) for the system. Then, using the computed control inputs and the noise-free dynamics, the sequence of states traversed  $\bar{\mathbf{x}}_{0:T}$  can be calculated.

*Design of feedback policy:* In order to design the LQR controller, the system is first linearised about the nominal trajectory  $(\bar{\mathbf{x}}_{0:T}, \bar{\mathbf{u}}_{0:T-1})$ . Using the linear time-varying system, the feedback policy is determined by minimizing a quadratic cost as shown in (6). The linear quadratic stochastic control problem (6) can be easily solved using the Riccati equation and the resulting policy is  $\delta \mathbf{u}_t = -\mathbf{L}_t \delta \mathbf{x}_t^\ell$ . The feedback gain and the Riccati equations are given by  $\mathbf{L}_t = (\mathbf{R} + \mathbf{B}_t^T \mathbf{P}_{t+1} \mathbf{B}_t)^{-1} \mathbf{B}_t^T \mathbf{P}_{t+1} \mathbf{A}_t$  and  $\mathbf{P}_t = \mathbf{A}_t^T \mathbf{P}_{t+1} \mathbf{A}_t - \mathbf{A}_t^T \mathbf{P}_{t+1} \mathbf{B}_t \mathbf{L}_t + \mathbf{Q}$ , respectively where  $\mathbf{Q}_f, \mathbf{Q} \succeq \mathbf{0}, \mathbf{R} \succ \mathbf{0}$  are the weight matrices for states and control and the terminal condition is  $\mathbf{P}_T = \mathbf{Q}_f$ .

#### 4.5 T-LQR with Replanning (T-LQR2):

T-LQR performs well at low noise levels, but at medium and high noise levels the system tends to deviate from the nominal. So, we define a threshold  $J_{\text{thresh}} = \frac{J_{0:t} - \bar{J}_{0:t}}{\bar{J}_{0:t}}$ , where  $J_{0:t}$  denotes the actual cost during execution till time  $t$ . The factor  $J_{\text{thresh}}$  measures the percentage deviation of the online trajectory from the nominal, and replanning is triggered for the system from the current state for the remainder of the horizon if the deviation exceeds it. Other replanning criteria such as state deviation can also be considered but we stick to the cost deviation in the following <sup>1</sup>. Note that if we set  $J_{\text{thresh}} = 0$ , T-LQR2 reduces to MPC. The calculation of the new nominal trajectory and LQR gains are carried out similarly to the explanation in Section 4.4. A generic algorithm for T-LQR and T-LQR2 is shown in Algorithm 1. The implementations of all the algorithms are available at [https://github.com/MohamedNaveed/Stochastic\\_Optimal\\_Control\\_algos](https://github.com/MohamedNaveed/Stochastic_Optimal_Control_algos).

#### 4.6 Multi-Agent versions

The MPC version of the multi-agent planning problem is reasonably straightforward except that the complexity of the planning increased (exponentially) in the number of agents. Also, we note that the agents have to always communicate with each other in order to do the planning. The Multi-agent Trajectory-optimised LQR (MT-LQR) version is also relatively straightforward in that the agents plan

<sup>1</sup> In the absence of a running cost, a criterion such as state deviation could be used. Since we aim to optimize the cost, a criterion based on cost seems more reasonable.

**Algorithm 1:** T-LQR2 algorithm

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**Input:** initial state  $\mathbf{x}_0$ , final state  $\mathbf{x}_g$ , time horizon  $T$ , replan threshold  $J_{\text{thresh}}$ , time step  $\Delta t$ , system and environment parameters  $\mathcal{P}$ .

- 1 **Function**  $Plan(\mathbf{x}_0, \mathbf{x}_g, T, \mathbf{u}_{init}, \mathbf{u}_{guess}, \mathcal{P})$  **is**
- 2      $\bar{\mathbf{u}}_{0:T-1} \leftarrow \text{OCP}(\mathbf{x}_0, \mathbf{x}_g, T, \mathbf{u}_{init}, \mathbf{u}_{guess}, \mathcal{P})$
- 3      $\bar{\mathbf{x}}_{t+1} \leftarrow f(\bar{\mathbf{x}}_t) + \mathbf{B}_t \bar{\mathbf{u}}_t; \quad t = 0, 1, \dots, T-1.$
- 4      $\mathbf{L}_{0:T-1} \leftarrow \text{Compute\_LQR\_Gain}(\bar{\mathbf{x}}_{0:T-1}, \bar{\mathbf{u}}_{0:T-1})$
- 5     return  $\bar{\mathbf{x}}_{0:T}, \bar{\mathbf{u}}_{0:T-1}, \mathbf{L}_{0:T-1}$
- 6 **end**
- 7 **Function**  $Main()$  **is**
- 8      $\bar{\mathbf{x}}_{0:T}, \bar{\mathbf{u}}_{0:T-1}, \mathbf{L}_{0:T-1} \leftarrow Plan(\mathbf{x}_0, \mathbf{x}_g, T, \mathbf{0}, \mathbf{u}_{guess}, \mathcal{P})$
- 9     **for**  $t \leftarrow 0$  **to**  $T-1$  **do**
- 10          $\mathbf{u}_t \leftarrow \text{Constrain}(\bar{\mathbf{u}}_t - \mathbf{L}_t(\mathbf{x}_t - \bar{\mathbf{x}}_t))$                                      // Enforce limits
- 11          $\mathbf{x}_{t+1} \leftarrow f(\mathbf{x}_t) + \mathbf{B}_t(\mathbf{u}_t + \epsilon \mathbf{w}_t)$
- 12         **if**  $(J_{0:t} - \bar{J}_{0:t})/\bar{J}_{0:t} > J_{\text{thresh}}$  **then**                                     // Replan?
- 13              $\bar{\mathbf{x}}_{t+1:T}, \bar{\mathbf{u}}_{t+1:T-1}, \mathbf{L}_{t+1:T-1} \leftarrow Plan(\mathbf{x}_{t+1}, \mathbf{x}_g, T-t-1, \mathbf{u}_t, \mathbf{u}_{guess}, \mathcal{P})$
- 14         **end**
- 15     **end**
- 16 **end**

---

the nominal path jointly once, and then the agents each track their individual paths using their decoupled feedback controllers. There is no communication whatsoever between the agents during this operation.

The MT-LQR2 version is a little more subtle. The agents have to periodically replan when the total cost deviates more than  $J_{\text{thresh}}$  away from the nominal, i.e., the agents do not communicate until the need to replan arises. In general, the system would need to detect this in a distributed fashion, and trigger replanning.

## 5 Simulation Results:

We test the performance of the algorithms extensively in three different non-linear models namely, the car-like robot model, car with two trailers and a quadrotor. Due to space constraints, only the results for the car-like robot are shown below, however, that the trends are generalizable can be seen from the results on the other models that are shown in the supplementary material. Numerical optimization is carried out using `CasADi` framework [1] with `Ipopt` [24] NLP solver in `Python`. To provide a good estimate of the performance, the results presented were averaged from 100 simulations for every value of noise considered. Simulations were carried out in parallel across 100 cores in a cluster equipped with Intel Xeon 2.5GHz E5-2670 v2 10-core processors.

### Car-like robot model:

The car-like robot considered in our work has the motion model described by  $x_{t+1} = x_t + v_t \cos(\theta_t) \Delta t$ ,  $y_{t+1} = y_t + v_t \sin(\theta_t) \Delta t$ ,  $\theta_{t+1} = \theta_t + \frac{v_t}{L} \tan(\phi_t) \Delta t$ ,  $\phi_{t+1} =$

$\phi_t + \omega_t \Delta t$ , where  $(x_t, y_t, \theta_t, \phi_t)^\top$  denote the robot’s state vector namely, robot’s  $x$  and  $y$  position, orientation and steering angle at time  $t$ . Also,  $(v_t, \omega_t)^\top$  is the control vector and denotes the robot’s linear velocity and angular velocity (i.e., steering). Here  $\Delta t$  is the discretization of the time step.

### Noise characterization:

We add zero mean independent identically distributed (i.i.d), random sequences ( $\mathbf{w}_t$ ) as actuator noise to test the performance of the control scheme. The standard deviation of the noise is  $\epsilon$  times the maximum value of the corresponding control input, where  $\epsilon$  is a scaling factor which is varied during testing, that is:  $\mathbf{w}_t = \mathbf{u}_{\max} \boldsymbol{\nu}$ ;  $\boldsymbol{\nu} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$  and the noise is added as  $\epsilon \mathbf{w}_t$ . Note that, we enforce the constraints in the control inputs before the addition of noise, so the controls can even take a value higher after noise is added. The analyses can be done with process noise as well, but  $\epsilon$  loses meaning in such a scenario and the plots would just shift depending on the variance of  $\mathbf{w}_t$ . Since all the algorithms use the same noise model and having been tested in an extensive range of values, the requirement for a process noise model is not really necessary.

### 5.1 Single agent setting:

A car-like robot is considered and is tasked to move from a given initial pose to a goal pose. The environment of the robot is shown in Figure 4. The experiment is done for all the control schemes discussed and their performance for different levels of noise are shown in Figure 2.

### 5.2 Multi-agent setting:

A labelled point-to-point transition problem with 3 car-like robots is considered where each agent is assigned a fixed destination which cannot be exchanged with another agent. The performance of the algorithms is shown in Figure 3. The cost function involves the state and control costs for the entire system similar to the single agent case. One major addition to the cost function is the penalty function to avoid inter-agent collisions which is given by  $\Psi^{(i,j)} = M \exp\left(-(\|\mathbf{p}_t^i - \mathbf{p}_t^j\|_2^2 - r_{\text{thresh}}^2)\right)$  where  $M > 0$  is a scaling factor,  $\mathbf{p}_t^i = (x_t^i, y_t^i)$  and  $r_{\text{thresh}}$  is the desired minimum distance the agents should keep between themselves.

### 5.3 Definition of noise regimes and discussion of the results:

Here we go on to define what exactly we mean by low, medium and high noise. The low noise regime as labelled in Figure 2b and 3b is the noise level at which the decoupled feedback law (T-LQR and MT-LQR) shows near-optimal performance compared to MPC and does not require any replanning operations. Beyond a limit replanning (T-LQR2 and MT-LQR2) is essential to constrain the cost from deviating away from the optimal and we call this as the medium noise regime.

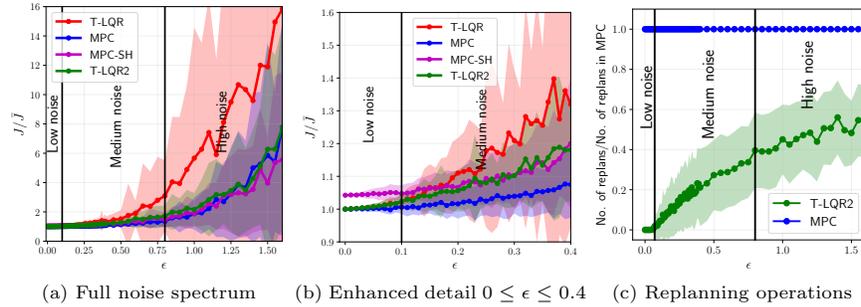


Fig. 2: Cost evolution of the different algorithms for varying noise for a single agent. Control Horizon ( $H_c$ ) used for MPC-SH was 7.  $J_{\text{thresh}} = 2\%$  was the replanning threshold used.  $J/\bar{J}$  is the ratio of the cost incurred during execution to the nominal cost and is used as the performance measure throughout the paper. The nominal cost  $\bar{J}$  which is calculated by solving the deterministic OCP for the total time horizon, just acts as a normalizing factor here. The solid line in the plots indicates the mean and the shade indicates the standard deviation of the corresponding metric.

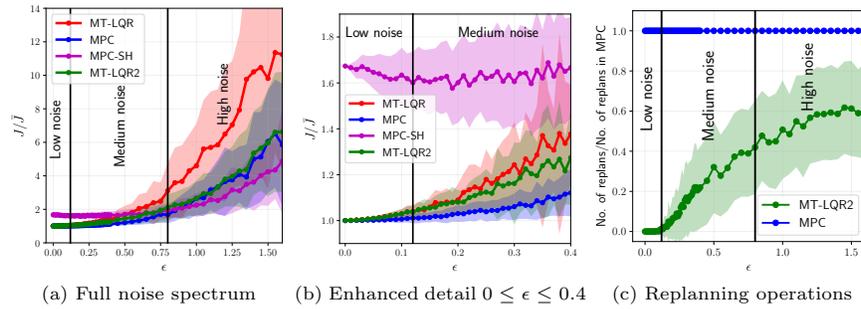


Fig. 3: Cost evolution of the different algorithms for varying noise for 3 agents. Control Horizon ( $H_c$ ) used for MPC-SH was 7.  $J_{\text{thresh}} = 2\%$  was the replanning threshold used.

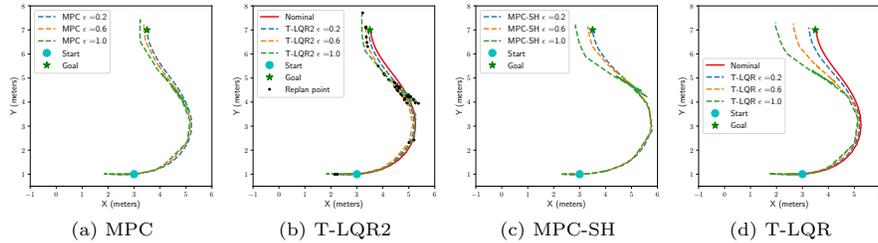
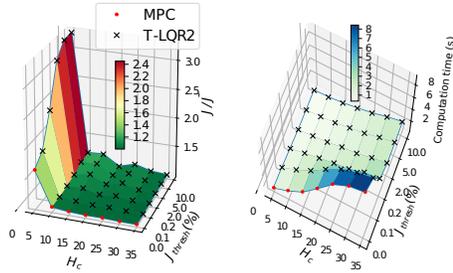
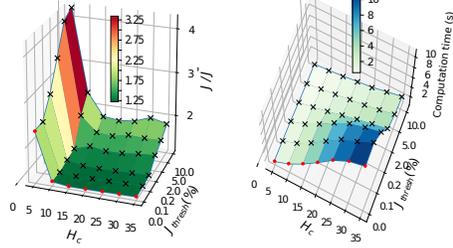


Fig. 4: The figure shows the paths taken by the robot using a particular algorithm and how they change as  $\epsilon$  varies. The key takeaway is that the paths taken under T-LQR are close to MPC for small values of  $\epsilon$  and deviate as  $\epsilon$  increases, while those of T-LQR2 are very close to the paths taken under MPC. It validates our claim about near-optimal performance of the decoupled feedback law under small noises and how performance can be preserved by replanning whenever necessary under medium noises. A equivalent plot for a scenario with obstacles is shown in the supplementary material.



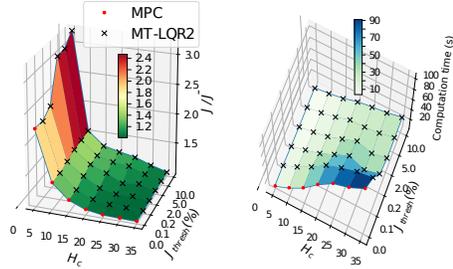
(a) Solution quality for 1 agent with  $\epsilon = 0.1$ .

(b) Compute time for 1 agent with  $\epsilon = 0.1$ .



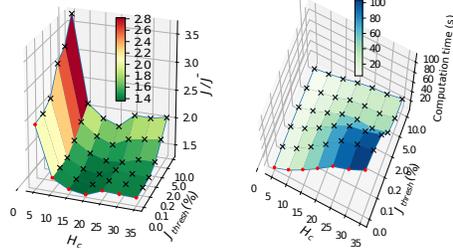
(c) Solution quality for 1 agent with  $\epsilon = 0.7$ .

(d) Compute time for 1 agent with  $\epsilon = 0.7$ .



(e) Solution quality for 3 agents with  $\epsilon = 0.1$ .

(f) Compute time for 3 agents with  $\epsilon = 0.1$ .



(g) Solution quality for 3 agents with  $\epsilon = 0.7$ .

(h) Compute time for 3 agents with  $\epsilon = 0.7$ .

Fig. 5: We study the variation seen in cost incurred and computation time by changing the  $J_{\text{thresh}}$  and control horizon ( $H_c$ ) in T-LQR2/MT-LQR2 and MPC for a single agent and a case with 3 agents. Figures 5a, 5b, 5c, 5d show for a single agent and 5e, 5f, 5g, 5h for 3 agents. 5a and 5b show the performance in terms of cost and computation time, respectively, for the same experiment at  $\epsilon = 0.1$  (low noise). Similarly, (c) and (d) show for  $\epsilon = 0.7$  (medium noise). To put into context, T-LQR2/MT-LQR2 replans less as  $J_{\text{thresh}}$  is increased which in turn leads to decrease in computation time. The computation time also decreases with decrease in  $H_c$ . Though MPC does not have a threshold for replanning, it is plotted at  $J_{\text{thresh}} = 0\%$  since it replans at every time step. We desire the performance indices, cost ( $J/J$ ) and computation time to be small.

As seen from 5a, for a fixed  $H_c$ , the performance of T-LQR2 is same as MPC and does not degrade as  $J_{\text{thresh}}$  is increased (except when  $H_c$  is too small). 5b shows the computational savings, where T-LQR2 is much better than MPC. From 5a and 5b it can be inferred that the decoupled approach provides good performance with substantial savings in computation time. Similarly, 5c shows that T-LQR2 performance is near-optimal to MPC for small  $J_{\text{thresh}}$  and degrades as it is relaxed, which indicates the necessity of replanning to maintain optimality in the medium noise regime. The corresponding variation in computation time is shown in 5d, where T-LQR2 does slightly better. The computational savings can be increased further but by trading away optimality.

Similar interpretation can be made for the multi-agent case as seen from Figures 5e, 5f, 5g, 5h. It can be seen that the positives of using the decoupled approach are amplified in the multi-agent case.

Now, we analyse how  $H_c$  affects the performance. Decreasing  $H_c$  leads to greedy sub-optimal solutions. Though not clearly seen in single agent case, the decrease in performance is seen in the multi-agent case (Fig. 5e). But in high noises, decreasing the horizon does not lead to decrease in solution quality or sometimes even produces better solutions as seen in 5g.

Figure 2c and 3c show the significant difference in the number of replanning operations, which determines the computational effort, taken by the decoupled approach compared to MPC. The significant difference in computational time between MPC and T-LQR2 can be seen from Figure 5b. The trend is similar in the multi-agent case which again shows that the decoupling feedback policy is able to give computationally efficient solutions which are near-optimal in low noise cases by avoiding frequent replanning.

In the high noise regime, T-LQR2, MT-LQR2 and even MPC-SH perform on a par with MPC as seen from Figures 2a and 3a meaning, planning too far ahead is not beneficial at high noise levels. It can also be seen in Figure 5g that the performance for MPC as well as MT-LQR2 is best at  $H_c = 20$ . Planning for a shorter horizon also eases the computation burden as seen in Figure 5h. It can also be seen in Figure 3a where MPC-SH with  $H_c = 7$  outperforms MPC with  $H_c = 35$  at high noise levels which again show that the effective planning horizon decreases at the high noise regime.

## 6 Conclusions and Implications

In this paper, we have considered a class of stochastic motion planning problems for robotic systems over a wide range of uncertainty conditions parameterized in terms of a noise parameter  $\epsilon$ . We have shown extensive empirical evidence that a simple generalization of a recently developed “decoupling principle” can lead to tractable planning without sacrificing performance for a wide range of noise levels. Future work will seek to treat the medium and high noise systems, considered here, analytically, and look to establish the near-optimality of the replanning scheme. Further, we shall consider the question of “when and how to replan” in a distributed fashion in the multi-agent setting, as well as relax the requirement of perfect state observation. It is also conjectured that by designing the linear feedback in a suitable fashion, the decoupling result can be made  $O(\epsilon^4)$  near-optimal, thus making the algorithm theoretically as good as MPC owing to Fleming’s result [8]. Further, an important limitation of the method is the smoothness of the nominal trajectory such that suitable Taylor expansions are possible, this breaks down when trajectories are non-smooth such as in hybrid systems like legged robots, or maneuvers have kinks for car-like robots such as in a tight parking application. It remains to be seen as to if, and how, one may extend the decoupling to such applications.

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